Carrying out a similar analysis, we can show that below the piece of surface (3.5) in the domain [1] there is a domain [$1_n$] which corresponds to stable fixed points of the transformations $T$ and $T^n$.

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# REGIONS OF STABILITY IN A CASE CLOSE TO THE CRITICAL ONE

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A system of real differential equations

$$x_s^{\cdot} = \mu_s x_s - \lambda_s y_s + X_s(x, y), \qquad y_s^{\cdot} = \mu_s y_s + \lambda_s x_s + Y_s(x, y)$$
$$x \equiv (x_1, \ldots, x_n), \quad y \equiv (y_1, \ldots, y_n) \quad (s = 1, \ldots, n) \tag{0.1}$$

is considered. Here $\mu_s$ are small, positive real parts of the complex-conjugate roots of the characteristic equation. $X_s$ and $Y_s$ are holomorphic functions of $x_s$ and $y_s$, and their expansions begin with the terms of at least second order.

The definition of the stability of motion in the cases close to the critical ones given in [1] and the results of [2] extended to cover the case of $n$ pairs of pure imaginary roots are used to establish the regions of stability for the system (0.1).

**1.** To be able to apply the Kamenkov [3, 4] transformation to our study of the stability of the system (0.1) for all $\mu_s = 0$ in the nonresonant cases, we consider the equivalent problem on the stability of the system

$$\rho^{\cdot} = 2\rho^{k+1} R_0(z) + 2\rho^{k+2} R_1(z) + \ldots$$
$$z_s^{\cdot} = 2\rho^k z_s (z_1 R_{s1}^{(2)} + z_2 R_{s2}^{(2)} + \ldots + z_n R_{sn}^{(2)}) + \ldots$$
$$R_{sj}^{(2)} = R_s^{(2)} - R_j^{(2)}, \quad R_0 = \sum z_j R_j^{(2)}(z) \quad (z_1 + \ldots + z_n = 1, \ s = 1, \ldots, n) \tag{1.1}$$

We can use the Kamenkov [3] theorem on instability as the basis for asserting that the unperturbed motion is unstable if at least one, nontrivial, real solution of the system of equations

$$z_s z_j R_{sj}^{(2)} = 0 \tag{1.2}$$

contains $R_0 > 0$.

Thus stability may occur only if all real straight lines (1.2) contain $R_0 \leqslant 0$.

It was shown in [5] for $s = n$ and in [2] for $s = 3$ that the condition $R_0 < 0$ on (1.2) is necessary and sufficient for the asymptotic stability in the third order forms.

When the problem on stability of (1.1) is solved by the third order forms, the Liapunov function can be taken in the form [2]

$$V = \rho \exp [- Nu (z)] \tag{1.3}$$

Here $N$ is a positive number, and $u$ is an arbitrary, continuous and bounded function of $z_1, \ldots, z_n$.

The derivative of (1.3) will be a negative-definite function equal to

$$V' = 2\rho^{k+1} e^{-Nu} [R_0 - N \sum z_s z_j (R_{sj}^{(2)})^2 P_{sj}] + \ldots \tag{1.4}$$

provided that the function $u$ can be selected from

$$\partial u / \partial z_s - \partial u / \partial z_j = R_{sj}^{(2)} P_{sj} \tag{1.5}$$

Here $P_{sj}$ are positive, continuous and bounded functions of $z_1, \ldots, z_n$, nonvanishing or vanishing only on straight lines (1.2); they may also be simply positive constants.

Expressions for $u$ and $P_{sj}$ were obtained in [2] for the case $n = 3$. Similar expressions can also be obtained for $u$ and $P_{sj}$ when $n \geqslant 4$. Assuming that all $P_{sj} = 1$ in (1.5) we find, that the asymptotic stability in the third order forms requires, in addition to the condition that $R_0 < 0$ in the solutions of (1.2), that a solution of the following system of algebraic equations exists (for some chosen function $V$)

$$\frac{\partial R_{sn}^{(2)}}{\partial z_{s+k}} - \frac{\partial R_{s+k,n}^{(2)}}{\partial z_s} = 0 \quad (s, k = 1, \ldots, n-2;\ s + k \leqslant n - 1) \tag{1.6}$$

We note that the values of the coefficients of this system can be altered when the corresponding substitution is introduced [2].

The problem on stability need not be solved by third order forms similarly as it is not solved by linear terms; then the results of [5] are not applicable.

This occurs when the expression for $R_0$ is zero on (1.2), and forms of higher order must then be considered when solving the problem.

We have the following theorem.

Theorem 1.1. Let the forms $R_s^{(2)}$ be such that when conditions (1.6) hold, $R_0 < 0$ on some of the lines (1.2) and equal to zero on the remaining lines, but $R_0 < 0$ within the cones with arbitrarily small vertex angles (axes of these cones are those lines of (1.2) on which $R_0 = 0$). If in addition we find that on the lines, on which $R_0 = 0$, we have $R_1 < 0$ or, if $R_0 = 0$, $R_1 = 0$, ..., $R_{k-1} = 0$ but $R_k < 0$, then the unperturbed motion is asymptotically stable.

The proof of this theorem uses the Liapunov function of the form

$$V = \rho e^{-Nu(z)} + a_1 \rho^2 + \ldots + a_k \rho_k^{k+1}$$

where $a_1, \ldots, a_k$ are positive numbers. The proof is analogous to that given in [2].

**2.** Let us now construct the regions of stability for the systems of the form (0.1). The system (0.1) can be transformed in an analogous manner into

$$\rho^{\cdot} = 2\rho \sum_{i=1}^{n} \mu_i z_i + 2\rho^{k+1} R_0(z) + \ldots$$

$$z^{\cdot}_s = 2z_s \left[ \left( \mu_s - \sum_{i=1}^{n} \mu_i z_i \right) + \rho^k \sum_{j=1, j\neq s}^{n} z_j R_{sj}^{(2)} \right] + \cdots \qquad (2.1)$$

$$R_0 = \Sigma z_j R_j(z), \quad R_{sj}^{(2)} = R_s^{(2)} - R_j^{(2)} \quad (s = 1, \ldots, n)$$

We isolate the regions of stability using a Liapunov function of the form (1.3), assuming that asymptotic stability in the third order forms exists for all $\mu_s = 0$.

Then the derivative of $V$ can be written, by virtue of (2.1), as

$$V' = 2\rho e^{-Nu} \left\{ \Sigma\mu_i z_i - N\Sigma \frac{\partial u}{\partial z_s} z_s (\mu_s - \Sigma\mu_i z_i) + \right.$$

$$\left. + \rho \left[ R_0 - N\Sigma z_s z_j (R_{sj}^{(2)})^2 P_{sj} \right] \right\} + \rho^3 S \qquad (2.2)$$

It was shown in Sect. 1, functions $u$ and $P_{sj}$ can be chosen in such a manner, that

$$[R_0 - N\, \Sigma z_s z_j (R_{sj}^{(2)})^2 P_{sj}] \leqslant -\delta$$

Here $\delta$ is a positive number such that

$$\delta = \inf |R_0|$$

on the real lines (1.2).

Moreover, for the chosen $u$ we can find

$$\nu = \sup \left[ \Sigma \mu_i z_i - N\Sigma \frac{\partial u}{\partial z_s} z_s (\mu_s - \Sigma\mu_i z_i) \right] \quad (z_1 + \cdots + z_n = 1)$$

Now the inner boundary $\rho_*$ of the regions of stability (for small $\rho$) will be given by

$$\rho_* = \nu / \delta$$

with the accuracy of up to the fifth order terms.

The approximate estimate of the outer boundary of the regions of stability (for small $\rho$) is obtained for given $V$ and $V'$ in the usual manner.

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